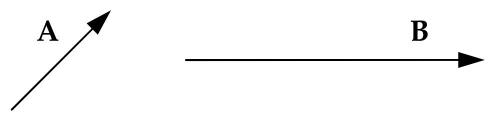
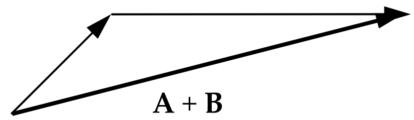
1. The Basics

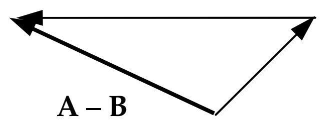
a) Given vectors **A** and **B** as shown, draw the following:



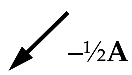
- **A** + **B**



- **A** – **B**



- –½ **A**

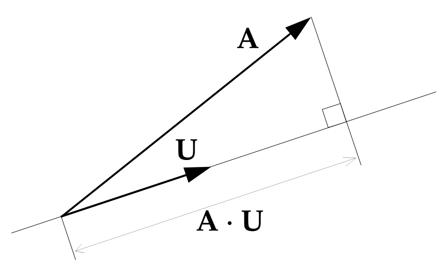


b) Write two equations for calculating the dot product $\mathbf{A} \cdot \mathbf{B}$, where $\mathbf{A} = [\ A_x\ A_y\ A_z\ ]$ and $\mathbf{B} = [\ B_x\ B_y\ B_z\ ]$.

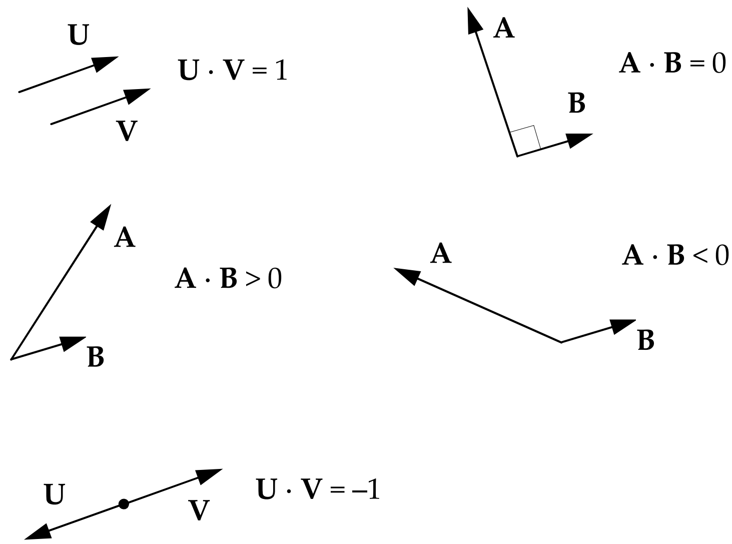
- $\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$

- $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \cos \theta$

c) Draw $\mathbf{A} \cdot \mathbf{U}$ on the diagram, given that $|\mathbf{U}| = 1$.

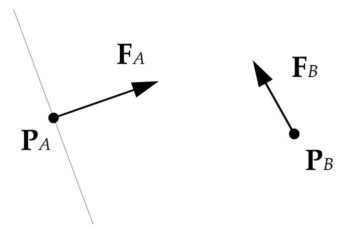


d) For each pair of vectors **A** and **B**, or **U** and **V**, write an inequality indicating the sign of the dot product… or if possible, write the exact value of the dot product. Note that $|\mathbf{U}| = |\mathbf{V}| = 1$, while $|\mathbf{A}| \neq 1$ and $|\mathbf{B}| \neq 1$.



2. Can you see me?

Two characters are standing on a roughly horizontal planar surface. The position of character A is $\mathbf{P}_A$ and its forward-facing unit vector is $\mathbf{F}_A$. Likewise the position and forward vector of character B are $\mathbf{P}_B$ and $\mathbf{F}_B$ respectively.



Top View

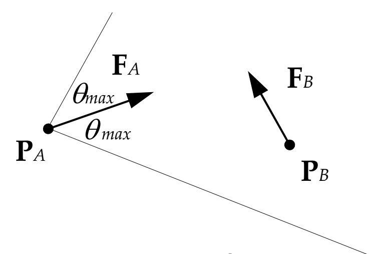
a) Use the sign of a dot product to determine whether character B is in front of or behind character A.

SOLUTION: Let $\mathbf{C} = \mathbf{P}_B - \mathbf{P}_A$.

If $(\mathbf{C} \cdot \mathbf{F}_A) \geq 0$ then character B is in front of A,

otherwise B is behind A.

b) Assume both characters have a vision cone extending θ_{max} radians to either side of their **F** vectors. Write an expression (using a dot product) indicating whether or not character A can "see" character B.
BONUS: How can we avoid finding the inverse cosine, $\cos^{-1}(\theta_{max})$?



Top View

SOLUTION: Let $\mathbf{C} = \mathbf{P}_B - \mathbf{P}_A$.

$(\mathbf{C} \cdot \mathbf{F}_A) = |\mathbf{C}| \cos \theta$, {recalling that $|\mathbf{F}_A| = 1$}

$\therefore\ \theta = \cos^{-1}((\mathbf{C} \cdot \mathbf{F}_A) / |\mathbf{C}|)$.

If $\theta \leq \theta_{max}$, then B can be seen by A.

or much more simply and less expensively…

If $((\mathbf{C} \cdot \mathbf{F}_A) / |\mathbf{C}|) \geq \cos\theta_{max}$, then B can be seen by A.

3. Ray Versus Sphere

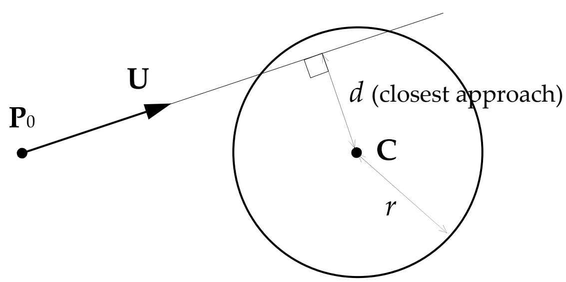
In many games, collision geometry is represented using spheres. Ray casts are a common way to query the collision world (e.g. line of sight queries, bullet traces, leg IK ground checks, etc.)

You are given a **sphere** defined by a center point **C** and a radius r. You are also given a **ray** defined by the parametric equation $\mathbf{P}(t) = \mathbf{P}_0 + t\mathbf{U}$, where $\mathbf{P}_0$ is the start point of the ray, **U** is a unit vector lying in the direction of the ray, and t is an arbitrary real-valued parameter in the range $[0, +\infty)$. Determine whether or not the ray intersects the sphere.

SOLUTION:

We are asked only to determine *if* the ray strikes the sphere, not *where*. As such, we simply need to check the ray's *closest approach* to the sphere's center point – i.e. the *perpendicular distance* from the ray to the sphere center. If this distance is less than or equal to the radius of the sphere, we have an intersection, otherwise we don't.

One complication is that we are testing a *ray*, not an infinite line. As such, we need to add two additional checks: First, we should check whether the initial point of the ray, $\mathbf{P}_0$, lies within the sphere—that would be an immediate intersection. Second, if the initial point is *not* inside the sphere, then we need to check that the point of closest approach occurs on the positive half of the ray… i.e. that the corresponding t value is positive.



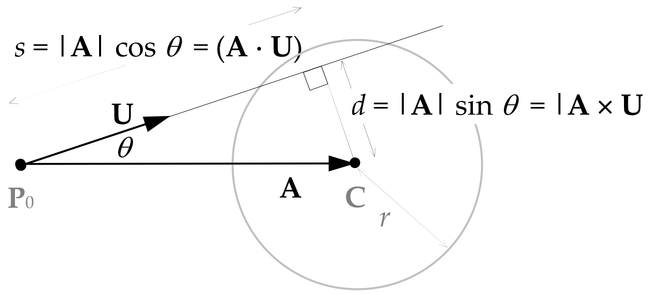
Initial Point Inside Sphere

Let vector $\mathbf{A} = \mathbf{C} - \mathbf{P}_0$.

If $(\mathbf{A} \cdot \mathbf{A}) \leq r^2$ then intersection has occurred.

Closest Approach

Otherwise, we must check the closest approach d against r. We must also check that the closest approach occurs within the half of the line that corresponds to the ray, which we'll do by finding the signed projection s of **A** onto **U**.



Noting that the dot and cross products are related to the cosine and sine of the angle between two vectors, respectively, we can find the components as follows:

$s = (\mathbf{A} \cdot \mathbf{U}) = |\mathbf{A}|\,|\mathbf{U}| \cos\theta = |\mathbf{A}| \cos\theta,$ because $|\mathbf{U}| = 1$

$d = |\mathbf{A} \times \mathbf{U}| = |\mathbf{A}|\,|\mathbf{U}| \sin\theta = |\mathbf{A}| \sin\theta.$

Then the ray intersects the sphere iff:

$d \leq r$ **and** $s \geq 0.$

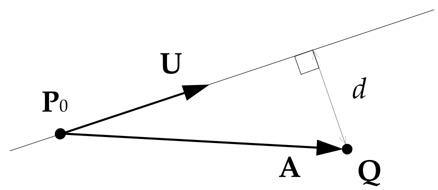
4. Wind Tunnel

The designers want to implement a shaft of wind that will affect any character or object that enters its cylindrical boundary.

a) You are given an arbitrary point **Q** in 3D space, and an infinite line represented by the locus of points $\mathbf{P}(t)$ defined as follows:

$$\mathbf{P}(t) = \mathbf{P}_0 + t\mathbf{U},$$

where $\mathbf{P}_0$ is a fixed point on the line, and **U** is a unit vector defining the line's direction. Find the perpendicular distance d from **Q** to the line.



SOLUTION: Let $\mathbf{A} = \mathbf{Q} - \mathbf{P}_0$.

Break into components parallel and perpendicular to **U**, respectively:

$\mathbf{A} = \mathbf{A}_{\parallel} + \mathbf{A}_{\perp}$.

Using the dot product, we have…

$\mathbf{A}_{\parallel} = (\mathbf{A} \cdot \mathbf{U})\mathbf{U}$, and then

$\mathbf{A}_{\perp} = \mathbf{A} - \mathbf{A}_{\parallel}$.

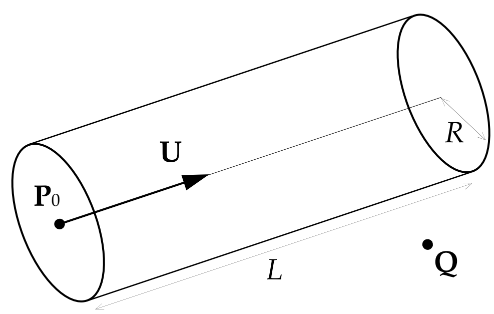
$\therefore\ d = |\mathbf{A}_{\perp}| = |\mathbf{A} - (\mathbf{A} \cdot \mathbf{U})\mathbf{U}|$.

or more simply…

$|\ \mathbf{A} \times \mathbf{U}\ | = |\mathbf{A}|\,|\mathbf{U}|\ \sin\theta = |\mathbf{U}|\ \sin\theta,$ remembering that $|\mathbf{U}| = 1$

$\therefore\ d = |\ \mathbf{A} \times \mathbf{U}\ |$.

b) The cylindrical wind tunnel can be defined by adding a radius r and length L to the infinite line from part (a). Assuming the position of our object or character is **Q**, write an expression that can be used to determine whether it will be affected by the wind or not.



SOLUTION: Let $\mathbf{A} = \mathbf{Q} - \mathbf{P}_0$,

let $s = |\mathbf{A}_{\parallel}| = (\mathbf{A} \cdot \mathbf{U})$, and

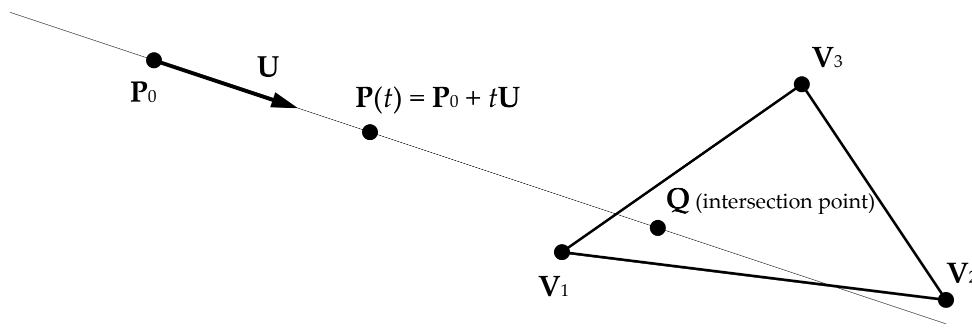
let $d = |\mathbf{A}_{\perp}| = |\mathbf{A} - \mathbf{A}_{\parallel}| = |\mathbf{A} - (\mathbf{A} \cdot \mathbf{U})\mathbf{U}| = |\mathbf{A} \times \mathbf{U}|$.

If $0 \le s \le L$ and $d \le R$, then point Q is affected by the wind, otherwise it isn't.

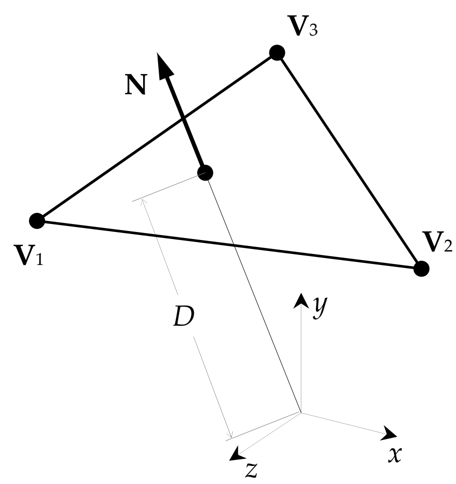
5. Ray Verus Triangle

Ray casts against triangles are also very common in games.

You are given a **triangle** defined by the three vertices $\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{V}_3$, and an **infinite line** defined by the parametric equation $\mathbf{P}(t) = \mathbf{P}_0 + t\mathbf{U}$, where $\mathbf{P}_0$ is any fixed point on the line, **U** is a unit vector lying in the direction of the line, and t is an arbitrary real-valued parameter. Determine if the **line intersects the triangle**, by following the three steps outlined below.



a) Find the equation of the plane, in the form $(\mathbf{N} \cdot \mathbf{P}) + D = 0$. (*i.e.* find **N** and D, given $\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{V}_3$.) Note that this is the same as writing $Ax + By + Cz + D = 0$, where $\mathbf{N} = (A, B, C)$ is the normal of the plane, D is the signed perpendicular distance from the plane to the origin, and $\mathbf{P} = (x, y, z)$ represents any arbitrary point on the plane.



SOLUTION:

We find the plane normal **N** by taking the cross product between any two edges, and normalizing the result:

$$\mathbf{N} = \frac{(\mathbf{V}_2 - \mathbf{V}_1) \times (\mathbf{V}_3 - \mathbf{V}_1)}{\left|(\mathbf{V}_2 - \mathbf{V}_1) \times (\mathbf{V}_3 - \mathbf{V}_1)\right|}.$$

Once we know **N**, we can find the perpendicular distance D from the plane to the origin by simply projecting *any point on the plane* onto **N**. We know *three* such points ($\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{V}_3$), so let's pick one:

$$(\mathbf{N} \cdot \mathbf{V}_1) + D = 0.$$

$$\therefore D = -(\mathbf{N} \cdot \mathbf{V}_1)$$
$$= -(\mathbf{N} \cdot \mathbf{V}_2)$$
$$= -(\mathbf{N} \cdot \mathbf{V}_3).$$

b) Find the point **Q** where the line $\mathbf{P}(t) = \mathbf{P}_0 + t\mathbf{U}$ intersects the plane $(\mathbf{N} \cdot \mathbf{P}) + D = 0$. How can you find the point **Q**, given your t?

SOLUTION:

Plug $\mathbf{P}(t)$ into the equation of the plane and solve for t.

$$(\mathbf{N} \cdot \mathbf{P}(t)) + D = 0$$
$$(\mathbf{N} \cdot (\mathbf{P}_0 + t\mathbf{U})) - (\mathbf{N} \cdot \mathbf{V}_1) = 0$$
$$(\mathbf{N} \cdot \mathbf{P}_0) + t(\mathbf{N} \cdot \mathbf{U}) - (\mathbf{N} \cdot \mathbf{V}_1) = 0$$
$$\therefore\ t_Q = \frac{\mathbf{N} \cdot \mathbf{V}_1 - \mathbf{N} \cdot \mathbf{P}_0}{\mathbf{N} \cdot \mathbf{U}}.$$

We can find **Q** by simply plugging t_Q into our original line equation.

$$\mathbf{Q} = \mathbf{P}_0 + t_Q\mathbf{U}.$$

c) We now know the intersection point **Q**, and that it lies on the plane. Determine whether it lies *inside* or *outside* the triangle. (The intersection only "counts" if **Q** is inside.)

Consider the edge between vertices $\mathbf{V}_1$ and $\mathbf{V}_2$. Now think about the two cases of **Q** being on the *inside* of this edge (toward the center of the triangle), versus it being on the *outside*. The cross product $\mathbf{C}_1 = (\mathbf{V}_2 - \mathbf{V}_1) \times (\mathbf{Q} - \mathbf{V}_1)$ differs in these two cases. This cross product is always perpendicular to the plane of the triangle, but it points in opposite *directions* depending on whether **Q** is inside or outside the triangle.

$$\mathbf{C}_1 = (\mathbf{V}_2 - \mathbf{V}_1) \times (\mathbf{Q} - \mathbf{V}_1)$$

$\mathbf{C}_2 = (\mathbf{V}_3 - \mathbf{V}_2) \times (\mathbf{Q} - \mathbf{V}_2)$

$\mathbf{C}_3 = (\mathbf{V}_1 - \mathbf{V}_3) \times (\mathbf{Q} - \mathbf{V}_3)$

(The order of the subtractions and cross products matter here!)

Now if all three cross products point in the *same direction*, then **Q** is inside the triangle; otherwise it is outside. We can test this quite easily by checking dot products:

If $(\mathbf{C}_1 \cdot \mathbf{C}_2) > 0$ and $(\mathbf{C}_1 \cdot \mathbf{C}_3) > 0$, then **Q** is inside the triangle.

NB: The three cross products $\mathbf{C}_i$ are related to the *Barycentric coordinates* of the point **Q**. Recall that a cross product $\mathbf{A}\times\mathbf{B}$ represents twice the *area* of the triangle formed by the vectors **A** and **B**. Hence the three cross products $\mathbf{C}_i$ are equal to twice the areas of the triangles formed by the three vertices and **Q**. If we divide each of these areas by the total area of the original triangle, we obtain the Barycentric coordinates (α, β, γ) of **Q**. Omitting the factors of 2 since they cancel out anyway, we have:

$\alpha = |(\mathbf{V}_2 - \mathbf{V}_1) \times (\mathbf{Q} - \mathbf{V}_1)| \;/\; |(\mathbf{V}_2 - \mathbf{V}_1) \times (\mathbf{V}_3 - \mathbf{V}_1)|$,

$\beta = |(\mathbf{V}_3 - \mathbf{V}_2) \times (\mathbf{Q} - \mathbf{V}_2)| \;/\; |(\mathbf{V}_2 - \mathbf{V}_1) \times (\mathbf{V}_3 - \mathbf{V}_1)|$,

$\gamma = |(\mathbf{V}_1 - \mathbf{V}_3) \times (\mathbf{Q} - \mathbf{V}_3)| \;/\; |(\mathbf{V}_2 - \mathbf{V}_1) \times (\mathbf{V}_3 - \mathbf{V}_1)|$.